Correlational properties of two-dimensional solvable chaos on the unit circle

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Abstract

This article investigates correlational properties of two-dimensional chaotic maps on the unit circle. We give analytical forms of higher-order covariances. We derive the characteristic function of their simultaneous and lagged ergodic densities. We found that these characteristic functions are described by three types of two-dimensional Bessel functions. Higher-order covariances between x and y and those between y and y show non-positive values. Asymmetric features between cosine and sine functions are elucidated.

1 Introduction

Knowledge on solvable chaos is useful for designing random number generators [1, 2, 3, 4] and Monte Carlo integration [5]. The idea of applying chaos theory to randomness has produced important works recently [6, 7, 8, 9]. Geisel and Fairen analyzed statistical properties of Chebyshev maps [10]. They showed the mixing properties and higher order moments with higher-order characteristic functions. González and Pino proposed a pseudo random number generator based on logistic maps [11]. Collins et al. [12] have applied the logit transformation to the logistic map variable for producing a sequence with a near Gaussian distribution. These solvable chaotic properties enable us to design and employ chaos for application purposes.

First, let us consider maps in the form of Chebyshev polynomials of degree k

$$x_{t+1} = T_k(t_t), \tag{1}$$

which map the interval $[-1, 1]$ onto the same interval. The first few polynomials are explicitly $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, and $T_3(x) = 4x^3 - 3x$. Since, there is permutability of the Chebyshev polynomials, $T_k(T_l(x)) = T_{kl}(x)$, Eq. (1) can be expressed as

$$x_t = T_{k^t}(x_0). \tag{2}$$

It was shown by Adler and Rivlin that Chebyshev maps with $k \geq 2$ are ergodic and strongly mixing. This map dynamics has the invariant measure $\mu(\mathrm{d}x) = \frac{\mathrm{d}x}{\pi\sqrt{1-x^2}}$. Geisel and Fairen shows that the characteristic function of the Chebyshev maps can be expressed as Bessel function [10]. They further considered the higher-order characteristic function. Following their strategy, we consider the characteristic function of two-dimensional solvable chaotic maps on a unit circle. We further calculate the higher-order covariance based on the characteristic function.

This article is organized as follows. In Sec. 2, we introduce two-dimensional chaotic maps on a unit circle. In Sec. 3, we show that simultaneous covariance among two variables is independent. In Sec. 4, we derive an analytical form of higher-order covariance among two variables. In Sec. 5, we compute higher-order covariance among two variables with lags. Sec. 6 is devoted to concluding remarks.

2 Two-dimensional solvable chaos

In this article, we consider two-dimensional maps on a unit circle. Suppose that $z_t = x_t + \sqrt{-1}y_t$ denotes a complex number, where x_t is a real number and y_t is an imaginary part at step $t \quad (t = 0, 1, \ldots)$. Then, we define the complex dynamics as

$$z_{t+1} = z_t^k, \tag{3}$$

where k is an integer. We can also express Eq. (3) as

$$\left\{ \begin{array}{lcl} x_{t+1} & = & P_k(x_t, y_t) \\ y_{t+1} & = & Q_k(x_t, y_t) \end{array} \right. , \tag{4}$$

where $P_k(x, y)$ and $Q_k(x, y)$ are defined as

$$(x + \sqrt{-1}y)^k = P_k(x, y) + \sqrt{-1}Q_k(x, y), \tag{5}$$

$$x^2 + y^2 = 1. \tag{6}$$

The first few polynomials are explicitly given by $P_1(x, y) = x$, $Q_1(x, y) = y$, $P_2(x, y) = x^2 - y^2$, $Q_2(x, y) = 2xy$, $P_3(x, y) = x^3 - 3xy^2$, $Q_3(x, y) = 3x^2y - y^3$, $P_4(x, y) = x^4 - 6x^2y^2 + y^4$, $Q_4(x, y) = 4x^3y - 4xy^3$, $P_5(x, y) = x^5 - 10x^3y^2 + 5xy^4$, $Q_5(x, y) = 5x^4y - 10x^2y^3 + y^5$, $P_6(x, y) = x^6 - 15x^4y^2 + 15x^2y^4 - y^6$, $Q_6(x, y) = 6x^5y - 20x^3y^3 + 6xy^5$, $P_7(x, y) = x^7 - 21x^5y^2 + 35x^3y^4 - 7xy^6$, and $Q_7(x, y) = 7x^6y - 35x^4y^3 + 21x^2y^5 - x^7$.

In general, $P_k(x, \pm\sqrt{1 - x^2}) = T_k(x)$ is satisfied. Specifically, $Q_k(x, y)$ for odd ordered k is equivalent to $Q_k(\pm\sqrt{1 - y^2}, y) = -T_k(y)$.

If we set an initial condition $z_0 = x_0 + \sqrt{-1}y_0$ on the unit circle $|z_0| = 1$, z_t is also mapped on the unit circle. In this case, Eq. (5) can be rewritten as

$$\exp(\sqrt{-1}\theta)^k = \exp(k\theta\sqrt{-1}), \tag{7}$$

where θ denotes the argument of (x, y) on the two-dimensional plane. It is convenient to represent the polynomial $P_k(x, y)$ and $Q_k(x, y)$ in the form

$$\begin{cases} P_k(\cos\theta, \sin\theta) & = & \cos(k\theta) \\ Q_k(\cos\theta, \sin\theta) & = & \sin(k\theta) \end{cases} \tag{8}$$

Fig. 1 shows a trajectory of (x_t, y_t) for $k = 2$. The value at each step stands on the unit circle.

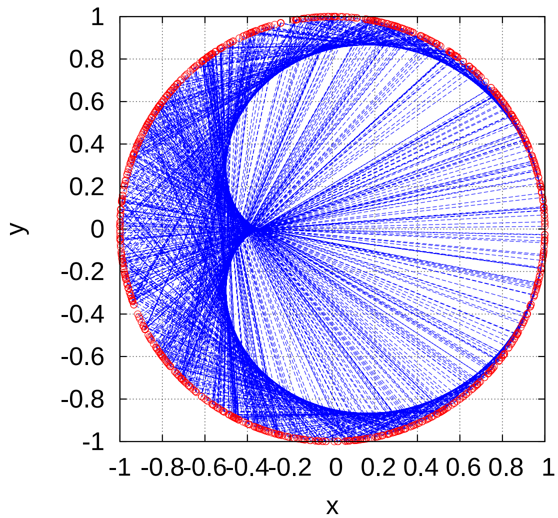

Figure 1: 800 steps of a trajectory of the two-dimensional chaotic map for $k = 2$. The initial value is given by $(x_0, y_0) = (-0.820000, 0.572364)$.

By introducing θ_t as the argument of z_t, we have

$$\theta_{t+1} = k\theta_t. \tag{9}$$

The solution of Eq. (9) can be written as

$$\theta_t = k^t \theta_0, \tag{10}$$

by using θ_0, denoted as the argument of z_0. Therefore, $z_t = x_t + \sqrt{-1}y_t$ is rewritten as

$$z_t = \cos(k^t\theta_0) + \sqrt{-1}\sin(k^t\theta_0) = \exp(k^t\theta_0\sqrt{-1}). \tag{11}$$

Eq. (10) is ergodic and has the constant invariant density $\rho_\Theta(\theta) = \frac{1}{2\pi} \quad (0 \leq \theta \leq 2\pi)$ since Eq. (9) is a Bernoulli map on mod 2π.

Transforming the orthogonal coordinate (x, y) into the polar coordinate (r, θ) by $x = r\cos\theta$ and $y = r\sin\theta$, we have $\rho_R(r) = \delta(r - 1)$. Therefore, the joint invariant density of x and y can be described as

$$\rho_{XY}(x,y) = \rho_\Theta(\theta)\rho_R(r)\left|\frac{\partial(\theta,r)}{\partial(x,y)}\right| = \frac{\delta(\sqrt{x^2+y^2}-1)}{2\pi\sqrt{x^2+y^2}}, \tag{12}$$

where $\delta(\cdot)$ represents Dirac's δ-function. The marginal density in terms of x is given by

$$\begin{aligned}
\rho_X(x) &= \int_{-1}^{1}\rho_{XY}(x,y)\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-1}^{1}\frac{\delta(\sqrt{x^2+y^2}-1)}{\sqrt{x^2+y^2}}\mathrm{d}y \\
&= \frac{1}{\pi}\int_{|x|-1}^{\sqrt{x^2+1}-1}\frac{\delta(t)}{\sqrt{(t+1)^2-x^2}}\mathrm{d}t \\
&= \frac{1}{\pi\sqrt{1-x^2}}.
\end{aligned}$$

In the same way, we obtain

$$\rho_Y(y) = \int_{-1}^{1}\rho_{XY}(x,y)\mathrm{d}x = \frac{1}{\pi\sqrt{1-y^2}}. \tag{13}$$

Note that $\rho_X(x)$ and $\rho_Y(y)$ are the same as the ergodic density of the Chebyshev maps.

3 Simultaneous covariance

Next, let us consider auto-correlations of x and y and cross-correlation between x and y. Obviously, mean values of x and y are given as zero.

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}x_t = \int_{-1}^{1}x\rho_X(x)\mathrm{d}x = \int_{-1}^{1}\frac{x}{\pi\sqrt{1-x^2}}\mathrm{d}x = 0, \tag{14}$$

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}y_t = \int_{-1}^{1}y\rho_Y(y)\mathrm{d}y = \int_{-1}^{1}\frac{y}{\pi\sqrt{1-y^2}}\mathrm{d}y = 0. \tag{15}$$

We shall introduce four types of correlations:

$$c_{XX}(\tau) = \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}x_t x_{t+\tau} = \int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y x\underbrace{P_k\circ\cdots\circ P_k}_{\tau}(x,y)\rho_{XY}(x,y) \tag{16}$$

$$c_{YY}(\tau) = \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}y_t y_{t+\tau} = \int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y y\underbrace{Q_k\circ\cdots\circ Q_k}_{\tau}(x,y)\rho_{XY}(x,y)$$

$$c_{XY}(\tau) = \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} x_t y_{t+\tau} = \int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y x \underbrace{Q_k\circ\cdots\circ Q_k}_{\tau}(x,y)\rho_{XY}(x,y) \tag{17}$$

$$c_{YX}(\tau) = \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} y_t x_{t+\tau} = \int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y y \underbrace{P_k\circ\cdots\circ P_k}_{\tau}(x,y)\rho_{XY}(x,y) \tag{18}$$

(19)

Transforming the orthogonal coordinate (x, y) into the polar coordinate (r, θ), we can calculate Eqs. (16) to (19) as

$$c_{XX}(\tau) = \frac{1}{2\pi}\int_0^{2\pi}\cos\theta\cos k^\tau\theta\mathrm{d}\theta = \frac{1}{2}\delta_{1,k^\tau} \tag{20}$$

$$c_{YY}(\tau) = \frac{1}{2\pi}\int_0^{2\pi}\sin\theta\sin k^\tau\theta\mathrm{d}\theta = \frac{1}{2}\delta_{1,k^\tau} \tag{21}$$

$$c_{XY}(\tau) = \frac{1}{2\pi}\int_0^{2\pi}\cos\theta\sin k^\tau\theta\mathrm{d}\theta = 0 \tag{22}$$

$$c_{YX}(\tau) = \frac{1}{2\pi}\int_0^{2\pi}\sin\theta\cos k^\tau\theta\mathrm{d}\theta = 0 \tag{23}$$

These are extensions of Chebyshev maps derived by Geisel and Fairen to the two-dimensional map [10]. Therefore, the auto-correlations of x and y decay 0 for $\tau \geq 1$, and the cross-correlations between x and y are zero. Furthermore, the correlation between z_t and $\overline{z_{t+\tau}}$, where $\bar{\cdot}$ is denoted as the complex conjugate of $\cdot$, is also zero,

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} z_t z_{t+\tau} = c_{XX}(\tau) - c_{YY}(\tau) + \sqrt{-1}\big(c_{XY}(\tau) + c_{YX}(\tau)\big) = 0. \tag{24}$$

Note that Eqs. (20) to (23) are derived by means of the permutability of z^k and the orthogonality between $P_k(x, y)$ and $Q_k(x, y)$. Clearly, from Eq. (3) we can prove the permutability of z^k such as $(z^k)^l = z^{kl}$. For $k \geq 1$ and $l \geq 1$, we also have the orthogonal relations among $P_k(x, y)$ and $Q_k(x, y)$

$$\int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y P_k(x,y)P_l(x,y)\rho_{XY}(x,y) = \frac{1}{2\pi}\int_0^{2\pi}\cos(k\theta)\cos(l\theta)\mathrm{d}\theta = \frac{1}{2}\delta_{k,l}, \tag{25}$$

$$\int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y Q_k(x,y)Q_l(x,y)\rho_{XY}(x,y) = \frac{1}{2\pi}\int_0^{2\pi}\sin(k\theta)\sin(l\theta)\mathrm{d}\theta = \frac{1}{2}\delta_{k,l}, \tag{26}$$

$$\int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y Q_k(x,y)P_l(x,y)\rho_{XY}(x,y) = \frac{1}{2\pi}\int_0^{2\pi}\sin(k\theta)\cos(l\theta)\mathrm{d}\theta = 0 \tag{27}$$

4 Simultaneous higher order covariance

Let us consider the characteristic function of the simultaneous joint density $\rho_{XY}(x,y)$, defined as

$$\begin{aligned}\Phi(u,v) &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} e^{\sqrt{-1}(ux_t+vy_t)} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\sqrt{-1}(ux+vy)}\rho_{XY}(x,y)\mathrm{d}x\mathrm{d}y. \qquad (28)\end{aligned}$$

Inserting Eq. (12) into Eq. (28), we have

$$\begin{aligned}\Phi(u,v) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\sqrt{-1}(ux+vy)}\frac{\delta(\sqrt{x^2+y^2}-1)}{\sqrt{x^2+y^2}}\mathrm{d}x\mathrm{d}y \\ &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\infty} r\mathrm{d}r e^{\sqrt{-1}(u\cos\theta+v\sin\theta)}\frac{\delta(r-1)}{2\pi r} \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{-1}(u\cos\theta+v\sin\theta)}\mathrm{d}\theta = J_0^{1,1}(u,v), \qquad (29)\end{aligned}$$

where $J_n^{p,q}(u,v)$ is defined as

$$J_n^{p,q}(u,v) = \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{-1}(u\cos(p\theta)+v\sin(q\theta)-n\theta)}\mathrm{d}\theta. \qquad (30)$$

This is similar to the two-dimensional Bessel function which was studied by Korsch et al. [13], however, it is a bit different from it. They define the two-dimensional Bessel functions with three integer indices n, p, and q as

$$\hat{J}_n^{p,q}(u,v) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\sqrt{-1}(u\sin(p\theta)+u\sin(q\theta)-n\theta)}\mathrm{d}\theta \qquad (31)$$

In his definition, the two-dimensional Bessel function consists of two sine functions. However, in our definition this consists of cosine and sine functions.

Clearly, both the two-dimensional Bessel functions satisfy

$$J_0^{1,1}(u,0) = J_0(u), \qquad J_0^{1,1}(0,v) = J_0(v), \qquad (32)$$
$$\hat{J}_0^{1,1}(u,0) = J_0(u), \qquad \hat{J}_0^{1,1}(0,v) = J_0(v), \qquad (33)$$

where $J_n(u)$ is the Bessel function defined as

$$J_n(u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\sqrt{-1}(n\theta-u\sin\theta)}\mathrm{d}\theta. \qquad (34)$$

In the one-dimensional case, Eq. (29) is equivalent to the characteristic function of Chebyshev polynomials, which is derived by Geisel and Fairen [10].

We can further expand $\Phi(u, v)$ in terms of u and v,

$$\begin{aligned}\Phi(u,v) &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(\sqrt{-1})^n}{n!}\int_0^{2\pi}(u\cos\theta+v\sin\theta)^n\mathrm{d}\theta \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(\sqrt{-1})^n}{n!}\sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right)u^m v^{n-m}\int_0^{2\pi}\cos^m\theta\sin^{n-m}\theta\mathrm{d}\theta.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\langle X^m Y^{n-m}\rangle &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}x_t^m y_t^{n-m} \\ &= \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y x^m y^{n-m}\rho_{XY}(x,y) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\cos^m\theta\sin^{n-m}\theta\mathrm{d}\theta. \quad (0\le m\le n). \end{aligned} \tag{35}$$

We also have the equality

$$\int_0^{\pi/2}\cos^{2p-1}\theta\sin^{2q-1}\theta\mathrm{d}\theta = \frac{1}{2}B(p,q) = \frac{1}{2}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{36}$$

where $B(a, b)$ denotes the beta function, defined as

$$B(a,b) = \int_0^1 \tau^{a-1}(1-\tau)^{b-1}\mathrm{d}\tau, \tag{37}$$

and $\Gamma(a)$ represents the gamma function, defined as

$$\Gamma(a) = \int_0^{\infty} e^{-\tau}\tau^{a-1}\mathrm{d}\tau. \tag{38}$$

Inserting Eq. (36) into $p = m/2 + 1/2$ and $q = (n - m)/2 + 1/2$ and using symmetry of cosine and sine functions and $\Gamma(n + 1) = n!$, we obtain

$$\langle X^m Y^{n-m}\rangle = \left\{\begin{array}{ll} \frac{2\Gamma(\frac{n-m+1}{2})\Gamma(\frac{m+1}{2})}{2\pi\Gamma(\frac{n}{2}+1)} = \frac{(m-1)!!(n-m-1)!!}{n!!} & (n, m : \text{even}) \\ 0 & (\text{otherwise}) \end{array}\right. . \tag{39}$$

Hence, the characteristic function of $\rho_{XY}(x, y)$ is described as

$$\Phi(u,v) = \sum_{n=0}^{\infty}(-1)^n\sum_{m=0}^{n}\frac{(u^2)^m(v^2)^{n-m}}{(2m)!!(2n-2m)!!(2n)!!}. \tag{40}$$

This is a natural extension of the Bessel function of degree 0 to the two-dimensional case,

$$J_0(z) = \sum_{r=0}^{\infty}\frac{(-z^2)^r}{(2r)!!(2r)!!}. \tag{41}$$

Since we can further calculate the m-th order moment of x_t and the $n-m$-th order moment of y_t as

$$\begin{aligned}\langle X^m\rangle &= \int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y x^m \rho_{XY}(x,y)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\cos^m\theta\mathrm{d}\theta = \begin{cases}\frac{(m-1)!!}{m!!} & (m:\text{even})\\ 0 & (m:\text{odd})\end{cases},\end{aligned} \tag{42}$$

and

$$\begin{aligned}\langle Y^{n-m}\rangle &= \int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y y^{n-m} \rho_{XY}(x,y)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\sin^{n-m}\theta\mathrm{d}\theta = \begin{cases}\frac{(n-m-1)!!}{(n-m)!!} & (n-m:\text{even})\\ 0 & (n-m:\text{odd})\end{cases},\end{aligned} \tag{43}$$

where $m!! = 2\cdot 4\cdot 6\cdots m$ for even m and $m!! = 1\cdot 3\cdot 5\cdots m$ for odd m, we get

$$\begin{aligned}\mathrm{Cov}[X^m, Y^{n-m}] &= \langle X^m Y^{n-m}\rangle - \langle X^m\rangle\langle Y^{n-m}\rangle\\ &= \begin{cases}\frac{(m-1)!!(n-m-1)!!}{n!!}\left[1-\frac{n!!}{m!!(n-m)!!}\right] & (m,n:\text{even})\\ 0 & (\text{otherwise})\end{cases}\end{aligned} \tag{44}$$

Here, we consider the negativity of even ordered moments. Hammersley suggested that antithetic variables are effective for variance reduction in Monte Carlo integrations [14]. The antithetic-variates method permits estimates through the use of negative correlated random variables faster than independent random variables. Let us confirm the sign of Eq. (44). We get

$$\begin{aligned}1-\frac{n!!}{m!!(n-m)!!} &= 1-\frac{(\frac{n}{2})!}{(\frac{m}{2})!(\frac{n-m}{2})!}\\ &= 1-\begin{pmatrix}\frac{n}{2}\\ \frac{m}{2}\end{pmatrix} \leq 0,\end{aligned} \tag{45}$$

since from the definition of combination, we have

$$\begin{pmatrix}\frac{n}{2}\\ \frac{m}{2}\end{pmatrix} = \frac{(\frac{n}{2})!}{(\frac{m}{2})!(\frac{n-m}{2})!} \geq 1. \tag{46}$$

The equality is satisfied if and only if $m=0$ or $m=n$. Note that Eq. (44) is independent of a value of k.

Therefore, Eq. (44) implies that x_t and y_t do not have any correlations for the odd-ordered moments, however, do have a negative covariance for the even-ordered moments. Fig. 2 shows the relationship between n and $\mathrm{Cov}[X^m, Y^{n-m}]$. It is confirmed that the covariance monotonically increases and approaches to zero as n increasing.

Furthermore, we calculate covariance between x_t^m and x_t^{n-m}, and between y_t^m and y_t^{n-m}. From Eqs. (42) and (43), we have

$$\begin{aligned}
\mathrm{Cov}[X^m, X^{n-m}] &= \mathrm{Cov}[Y^m, Y^{n-m}] \\
&= \begin{cases} \frac{1}{2^n}\Big[\begin{pmatrix} n \\ \frac{n}{2} \end{pmatrix} - \begin{pmatrix} m \\ \frac{m}{2} \end{pmatrix}\begin{pmatrix} n-m \\ \frac{n-m}{2} \end{pmatrix}\Big] \geq 0 & (n, m : \text{even}) \\ 0 & (\text{otherwise}) \end{cases} .
\end{aligned} \tag{47}$$

The non-negativity of Eq. (47) is proven as follows. Let us consider the case that n is even. From

$$(1+x)^n = \Big\{(1+x)^{\frac{n}{2}}\Big\}^2, \tag{48}$$

one has

$$\sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} x^m = \Big(\sum_{m=0}^{\frac{n}{2}} \begin{pmatrix} \frac{n}{2} \\ m \end{pmatrix} x^m\Big)^2 \tag{49}$$

Comparing x^m's coefficient, we get the following inequality

$$\begin{pmatrix} n \\ m \end{pmatrix} \geq \begin{pmatrix} \frac{n}{2} \\ \frac{m}{2} \end{pmatrix}^2 . \tag{50}$$

Therefore, we obtain

$$\frac{1}{2^n}\Big[\begin{pmatrix} n \\ \frac{n}{2} \end{pmatrix} - \begin{pmatrix} m \\ \frac{m}{2} \end{pmatrix}\begin{pmatrix} n-m \\ \frac{n-m}{2} \end{pmatrix}\Big] = \frac{1}{2^n}\frac{\begin{pmatrix} n \\ \frac{n}{2} \end{pmatrix}}{\begin{pmatrix} n \\ m \end{pmatrix}}\Big[\begin{pmatrix} n \\ m \end{pmatrix} - \begin{pmatrix} \frac{n}{2} \\ \frac{m}{2} \end{pmatrix}^2\Big] \geq 0. \tag{51}$$

5 Higher order covariance with lags

More generally, we can introduce a characteristic function of the joint density between x_{t+p}^m and y_{t+q}^{n-m}.

$$\begin{aligned}
\Psi_{XY}(u,v) &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} e^{\sqrt{-1}(ux_{t+p}+vy_{t+q})} \\
&= \Big\langle \exp\big(\sqrt{-1}(u\underbrace{P_k\circ\cdots\circ P_k}_{p}(x,y) + v\underbrace{Q_k\circ\cdots\circ Q_k}_{q}(x,y))\big)\Big\rangle \\
&= \int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y \exp\big(\sqrt{-1}(u\underbrace{P_k\circ\cdots\circ P_k}_{p}(x,y) + v\underbrace{Q_k\circ\cdots\circ Q_k}_{q}(x,y))\big)\rho_{XY}(x,y) \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{-1}(u\cos(k^p\theta)+v\sin(k^q\theta))}\mathrm{d}\theta = J_0^{k^p,k^q}(u,v).
\end{aligned} \tag{52}$$

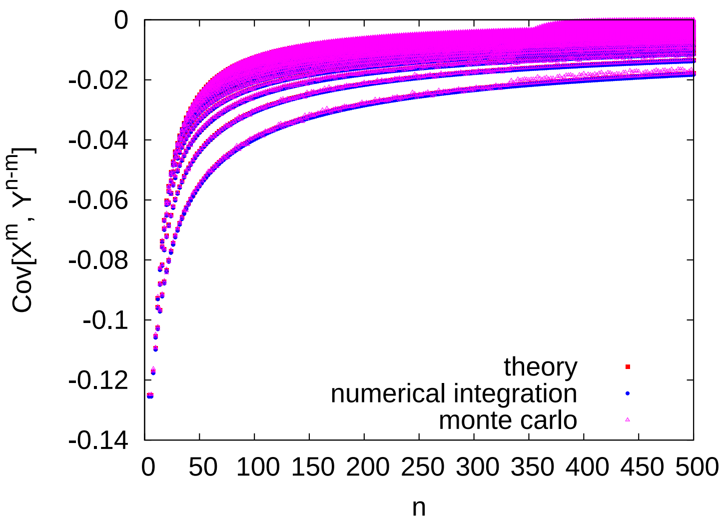

Figure 2: The relationship between n and $\mathrm{Cov}[X^m, Y^{n-m}]$ for $k = 2$.

Similarly to $\Phi(u, v)$, from the expansion in terms of u and v, we obtain

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} x_{t+p}^{m} y_{t+q}^{n-m} &= \int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y\,\underbrace{P_k\circ\cdots\circ P_k}_{p}(x,y)\,\underbrace{Q_k\circ\cdots\circ Q_k}_{q}(x,y)\rho_{XY}(x,y)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\cos^m(k^p\theta)\sin^{n-m}(k^q\theta)\mathrm{d}\theta. \qquad (53)\end{aligned}$$

By using

$$\begin{aligned}&\cos^m(k^p\theta)\sin^{n-m}(k^q\theta)\\ =\ &\frac{1}{2^m}(e^{\sqrt{-1}k^p\theta}+e^{-\sqrt{-1}k^p\theta})^m\frac{1}{(2\sqrt{-1})^{n-m}}(e^{\sqrt{-1}k^q\theta}-e^{-\sqrt{-1}k^q\theta})^{n-m}\\ =\ &\frac{1}{2^n(\sqrt{-1})^{n-m}}\sum_{r=0}^{m}\sum_{s=0}^{n-m}(-1)^{n-m-s}\frac{m!}{r!(m-r)!}\frac{(n-m)!}{s!(n-m-s)!}e^{\sqrt{-1}[(2r-m)k^p+(2s-n+m)k^q]\theta},\end{aligned} \qquad (54)$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{-1}\alpha\theta}\mathrm{d}\theta = \delta_{0,\alpha}, \qquad (55)$$

we obtain

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} x_{t+p}^{m} y_{t+q}^{n-m}$$

$$= \begin{cases} \frac{(-1)^{\frac{n-m}{2}}}{2^n} \sum_{r=0}^{m} \sum_{s=0}^{n-m} \frac{m!}{r!(m-r)!} \frac{(n-m)!}{s!(n-m-s)!} (-1)^{-s} \delta_{0,(2r-m)k^p+(2s-n+m)k^q} & (m, n : \text{even}) \\ 0 & (\text{otherwise}) \end{cases} . \tag{56}$$

Since we further have

$$\begin{aligned} \langle X_{t+p}^m \rangle &= \int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} \mathrm{d}y \Big[\underbrace{P_k \circ \cdots \circ P_k}_{p}(x, y) \Big]^m \rho_{XY}(x, y) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cos^m(k^p \theta) \mathrm{d}\theta \\ &= \begin{cases} \frac{(m-1)!!}{m!!} & (m : \text{even}) \\ 0 & (m : \text{odd}) \end{cases} , \end{aligned} \tag{57}$$

and

$$\begin{aligned} \langle Y_{t+q}^{n-m} \rangle &= \int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} \mathrm{d}y \Big[\underbrace{Q_k \circ \cdots \circ Q_k}_{q}(x, y) \Big]^{n-m} \rho_{XY}(x, y) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cos^{n-m}(k^q \theta) \mathrm{d}\theta \\ &= \begin{cases} \frac{(n-m-1)!!}{(n-m)!!} & (n-m : \text{even}) \\ 0 & (n-m : \text{odd}) \end{cases} , \end{aligned} \tag{58}$$

we get

$$\mathrm{Cov}[X_{t+p}^m, Y_{t+q}^{n-m}] = \langle X_{t+p}^m Y_{t+q}^{n-m} \rangle - \langle X_{t+p}^m \rangle \langle Y_{t+q}^{n-m} \rangle$$

$$= \begin{cases} \frac{(-1)^{\frac{n-m}{2}}}{2^n} \sum_{r=0}^{m} \sum_{s=0}^{n-m} \begin{pmatrix} m \\ r \end{pmatrix} \begin{pmatrix} n-m \\ s \end{pmatrix} (-1)^{-s} \delta_{0,(2r-m)k^p+(2s-n+m)k^q} \\ \qquad - \frac{(m-1)!!(n-m-1)!!}{m!!(n-m)!!} & (m, n : \text{even}) \\ 0 & (\text{otherwise}) \end{cases} . \tag{59}$$

Kohda et al. showed that the higher-order covariance of Chebyshev maps have no correlation [15]. We use their derivation in our case. According to Kac's statistical independence [16] when in Eq. (59)

$$(2r - m)k^p + (2s - n + m)k^q = 0, \quad (0 \le r \le m; 0 \le s \le n - m) \tag{60}$$

holds for any k^p and k^q if and only if $r = m/2$ and $s = (n - m)/2$, k^p and k^q are called linearly independent. Then x_{t+p}^m and y_{t+q}^{n-m} are statistically independent [15].

Let consider the case that m and n are even. From elementary facts about the theory of numbers, we know that

$$N = k^e + r \quad (0 \le r < k), \tag{61}$$

where N is a natural number, and k, e and r are non-negative integers. In the case that $2r-m>0$, $2s-n+m<0$, and $p<q$ we have

$$\begin{aligned}(2r-m)k^p+(2s-n+m)k^q &= \{(2r-m)+(2s-n+m)k^{q-p}\}k^p \\ &= \{(k^{e_1}+r')-(k^{e_2}+s')k^{q-p}\}k^p \\ &= (k^{e_1}+r'-k^{e_2+q-p}-s'k^{q-p})k^p. \quad (62)\end{aligned}$$

Therefore, if $[(2r-m)/k]=0$, $[(2s-n+m)/k]=0$, and $e_1=e_2+q-p$ hold then $(2r-m)k^p+(2s-n+m)k^q=0$ is satisfied for integers r and s other than $r=m/2$ and $s=(n-m)/2$. When $m>k$, and $n-m>k$, we have $[(2r-m)/k]=0$ and $[(2s-n+m)/k]=0$. Therefore, $m\geq k^{e_1}$ and $n-m\geq k^{e_2}$ would be satisfied. Namely, when $n<k^{e_1}+k^{e_2}=k^{e_2}(k^{q-p}+1)$, x^m_{t+p} and y^{n-m}_{t+q} are statistically independent. This implies that $q-p$ goes infinity, x^m_{t+p} and y^{n-m}_{t+q} become statistically independent in an exponential manner.

Fig. 3 shows $\mathrm{Cov}[X^m_{t+p}, Y^{n-m}_{t+q}]$ for $(p,q)=(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$, $(0,5)$, and $(0,6)$. As shown in figures, we found that the covariances decrease as $|p-q|$ increasing. The range of the covariances approach to zero as q increasing.

Obviously, Eq. (60) has solutions $r=m/2$ and $s=(n-m)/2$. A sum of the contributions for $r=m/2$ and $s=(n-m)/2$ in Eq. (71) is equivalent to $\frac{(m-1)!!(n-m-1)!!}{m!!(n-m)!!}$. Since $\mathrm{Cov}[X^m_{t+p}, Y^{n-m}_{t+q}]$ is less than zero from the numerical simulation, for solutions other than $r=m/2$ and $s=(n-m)/2$ of Eq. (60), it should satisfy that a sum of negative contributions is greater than a sum of positive contributions.

We may consider two types of second-order characteristic functions with lags. Note that Geisel and Fairen [10] considered a similar second-order characteristic function for the Chebyshev maps. Their characteristic function corresponds to $\Psi_{XX}(u,u)$ in our definition.

$$\begin{aligned}\Psi_{XX}(u,v) &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} e^{\sqrt{-1}(ux_{t+p}+vx_{t+q})} \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{-1}(u\cos(k^p\theta)+v\cos(k^q\theta))}\mathrm{d}\theta \quad (63)\\ \Psi_{YY}(u,v) &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} e^{\sqrt{-1}(uy_{t+p}+vy_{t+q})} \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{-1}(u\sin(k^p\theta)+v\sin(k^q\theta))}\mathrm{d}\theta \quad (64)\end{aligned}$$

Similarly to $\Psi_{XY}(u,v)$, from the expansion in terms of u and v, we obtain

$$\Psi_{XX}(u,v) = \sum_{n=0}^{\infty}\frac{(-1)^{\frac{n}{2}}}{n!}\sum_{m=0}^{n}\begin{pmatrix} n \\ m\end{pmatrix}\langle X^m_{t+p}X^{n-m}_{t+q}\rangle u^m v^{n-m}, \quad (65)$$

$$\Psi_{YY}(u,v) = \sum_{n=0}^{\infty}\frac{(-1)^{\frac{n}{2}}}{n!}\sum_{m=0}^{n}\begin{pmatrix} n \\ m\end{pmatrix}\langle Y^m_{t+p}Y^{n-m}_{t+q}\rangle u^m v^{n-m}, \quad (66)$$

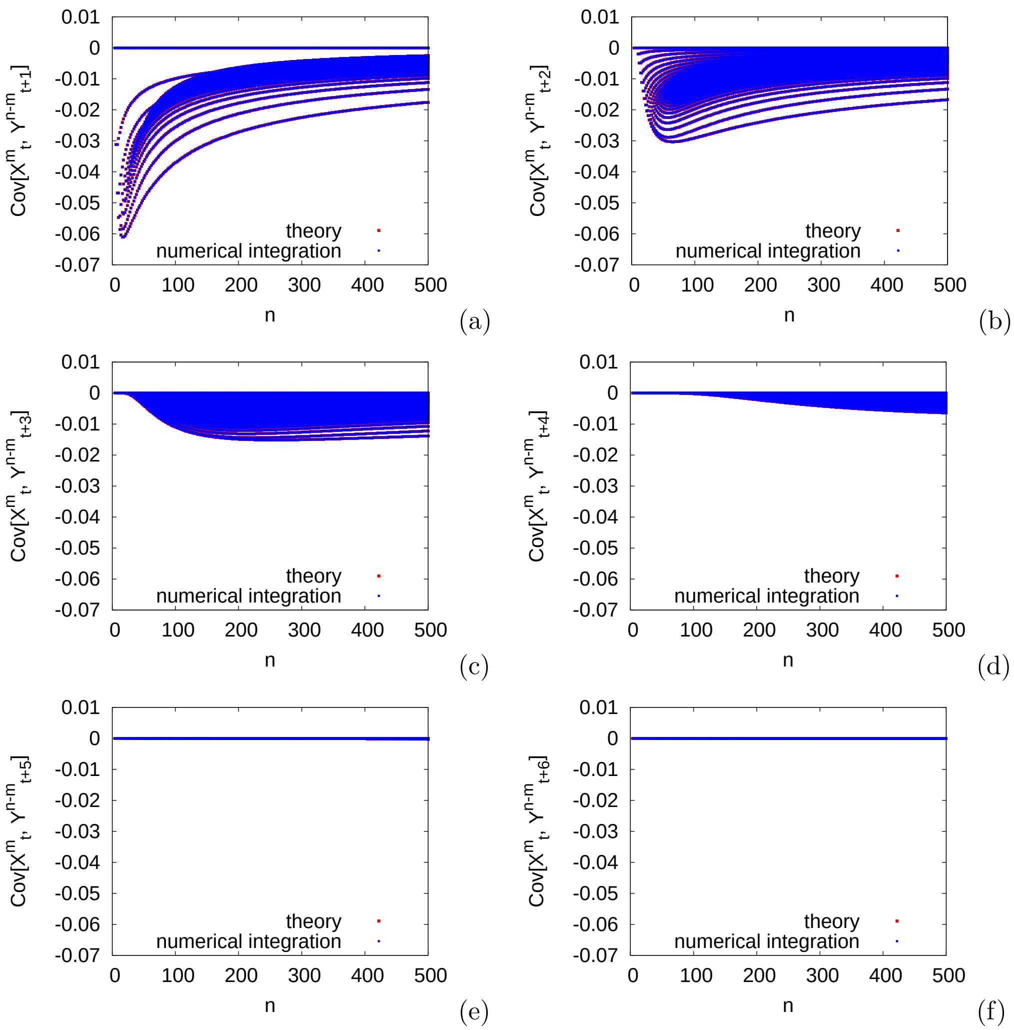

Figure 3: Scatter plots of $\mathrm{Cov}[X_{t+p}^{m}, Y_{t+q}^{n-m}]$ in terms of n ($0 \le m \le n$) at $k = 2$ and $p = 0$, (a) $q = 1$, (b) $q = 2$, (c) $q = 3$, (d) $q = 4$, (e) $q = 5$, and (f) $q = 6$. Filled squares represent theoretical values, and filled circles values obtained from numerical integration.

where

$$\langle X_{t+p}^{m} X_{t+q}^{n-m} \rangle = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} x_{t+p}^{m} x_{t+q}^{n-m} = \frac{1}{2\pi} \int_0^{2\pi} \cos^m(k^p\theta)\cos^{n-m}(k^q\theta)\mathrm{d}\theta,$$

$$\langle Y_{t+p}^{m} Y_{y+q}^{n-m} \rangle = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} y_{t+p}^{m} y_{t+q}^{n-m} = \frac{1}{2\pi} \int_0^{2\pi} \sin^m(k^p\theta)\sin^{n-m}(k^q\theta)\mathrm{d}\theta.$$

By using

$$
\begin{aligned}
& \cos^m(k^p\theta)\cos^{n-m}(k^q\theta) \\
= \; & \frac{1}{2^m}(e^{\sqrt{-1}k^p\theta}+e^{-\sqrt{-1}k^p\theta})^m\frac{1}{2^{n-m}}(e^{\sqrt{-1}k^q\theta}+e^{-\sqrt{-1}k^q\theta})^{n-m} \\
= \; & \frac{1}{2^n}\sum_{r=0}^{m}\sum_{s=0}^{n-m}\frac{m!}{r!(m-r)!}\frac{(n-m)!}{s!(n-m-s)!}e^{\sqrt{-1}[(2r-m)k^p+(2s-n+m)k^q]\theta}, \\
& \sin^m(k^p\theta)\sin^{n-m}(k^q\theta) \\
= \; & \frac{1}{2\sqrt{-1})^m}(e^{\sqrt{-1}k^p\theta}-e^{-\sqrt{-1}k^p\theta})^m\frac{1}{(2\sqrt{-1})^{n-m}}(e^{\sqrt{-1}k^q\theta}-e^{-\sqrt{-1}k^q\theta})^{n-m} \\
= \; & \frac{(-1)^{\frac{n}{2}}}{2^n}\sum_{r=0}^{m}\sum_{s=0}^{n-m}\frac{m!}{r!(m-r)!}\frac{(n-m)!}{s!(n-m-s)!}(-1)^{-r-s}e^{\sqrt{-1}[(2r-m)k^p+(2s-n+m)k^q]\theta},
\end{aligned}
$$

therefore, we have

$$
\lim_{T\to\infty}\tfrac{1}{T}\textstyle\sum_{t=0}^{T-1}x_{t+p}^m x_{t+q}^{n-m}= \\
\begin{cases} \frac{1}{2^n}\sum_{r=0}^{m}\sum_{s=0}^{n-m}\frac{m!}{r!(m-r)!}\frac{(n-m)!}{s!(n-m-s)!}\delta_{0,(2r-m)k^p+(2s-n+m)k^q} & (m,n:\text{even}) \\ 0 & (\text{otherwise}) \end{cases}, \tag{67}
$$

$$
\lim_{T\to\infty}\tfrac{1}{T}\textstyle\sum_{t=0}^{T-1}y_{t+p}^m y_{t+q}^{n-m}= \\
\begin{cases} \frac{(-1)^{\frac{n}{2}}}{2^n}\sum_{r=0}^{m}\sum_{s=0}^{n-m}\frac{m!}{r!(m-r)!}\frac{(n-m)!}{s!(n-m-s)!}(-1)^{-r-s}\delta_{0,(2r-m)k^p+(2s-n+m)k^q} & (m,n:\text{even}) \\ 0 & (\text{otherwise}) \end{cases} \tag{68}
$$

We further have

$$
\begin{aligned}
& \mathrm{Cov}[X_{t+p}^m, X_{t+q}^{n-m}] = \langle X_{t+p}^m X_{t+q}^{n-m}\rangle - \langle X_{t+p}^m\rangle\langle X_{t+q}^{n-m}\rangle \\
= \; & \begin{cases} \frac{1}{2^n}\sum_{r=0}^{m}\sum_{s=0}^{n-m}\begin{pmatrix} m \\ r \end{pmatrix}\begin{pmatrix} n-m \\ s \end{pmatrix}\delta_{0,(2r-m)k^p+(2s-n+m)k^q} \\ \qquad -\left(\frac{(m-1)!!}{m!!}\right)^2 & (m,n:\text{even}) \\ 0 & (\text{otherwise}) \end{cases} \quad (69)
\end{aligned}
$$

(70)

A sum of contributions for $r = m/2$ and $s = (n-m)/2$ in Eq. (70) is equivalent to $(\frac{(m-1)!!}{m!!})^2$. If Eq. (60) has other solutions than $r = m/2$ and $s = (n-m)/2$, then the covariance positively increases. Therefore, we could prove $\mathrm{Cov}[X_{t+p}^m, X_{t+q}^{n-m}] \geq 0$.

We also have

$$
\begin{aligned}
& \mathrm{Cov}[Y_{t+p}^m, Y_{t+q}^{n-m}] = \langle Y_{t+p}^m Y_{t+q}^{n-m}\rangle - \langle Y_{t+p}^m\rangle\langle Y_{t+q}^{n-m}\rangle \\
= \; & \begin{cases} \frac{(-1)^{\frac{n}{2}}}{2^n}\sum_{r=0}^{m}\sum_{s=0}^{n-m}\begin{pmatrix} m \\ r \end{pmatrix}\begin{pmatrix} n-m \\ s \end{pmatrix}(-1)^{-r-s}\delta_{0,(2r-m)k^p+(2s-n+m)k^q} \\ \qquad -\left(\frac{(n-m-1)!!}{(n-m)!!}\right)^2 & (m,n:\text{even}) \\ 0 & (\text{otherwise}) \end{cases} \quad (71)
\end{aligned}
$$

Fig. 5 shows covariance between X^m_{t+p} and X^{n-m}_{t+q}, and between Y^m_{t+p} and Y^{n-m}_{t+q}. It is found that $\mathrm{Cov}[X^m_{t+p}, X^{n-m}_{t+q}]$ shows non-negative values, and that $\mathrm{Cov}[Y^m_{t+p}, Y^{n-m}_{t+q}]$ shows non-positive values. We found that $\mathrm{Cov}[X^m_{t+p}, Y^{n-m}_{t+q}]$ takes the same non-positive value as $\mathrm{Cov}[Y^m_{t+p}, Y^{n-m}_{t+q}]$ for $p \neq q$ from Figs. 3 and 5. The reason is because $\cos^m(k^p\theta)\sin^{n-m}(k^q\theta)$ and $\sin^m(k^p\theta)\sin^{n-m}(k^q\theta)$ have the same area to the x-axis, but $\cos^m(k^p\theta)\cos^{n-m}(k^q\theta)$ is different from them as shown in Fig. 4.

A sum of the contributions for $r = m/2$ and $s = (n-m)/2$ in Eq. (71) is equivalent to $(\frac{(n-m-1)!!}{(n-m)!!})^2$. Since $\mathrm{Cov}[Y^m_{t+p}, Y^{n-m}_{t+q}]$ is less than zero from the numerical simulation, for solutions other than $r = m/2$ and $s = (n-m)/2$ of Eq. (60), it should satisfy that a sum of negative contributions is greater than a sum of positive contributions.

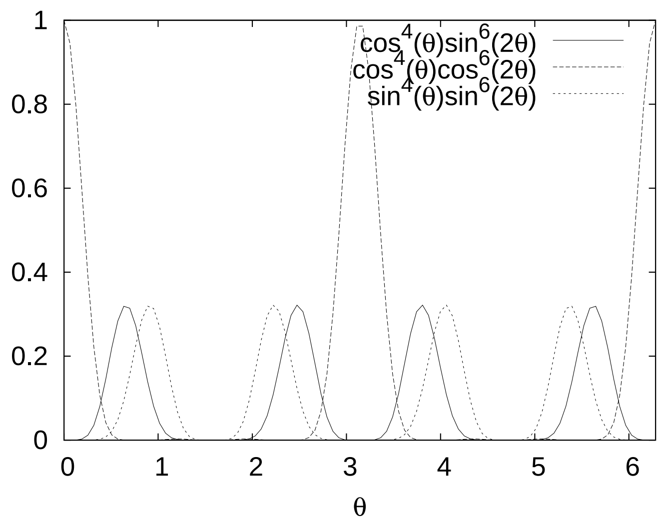

Figure 4: The wave forms of $\cos^m(k^p\theta)\sin^{n-m}(k^q\theta)$, $\sin^m(k^p\theta)\sin^{n-m}(k^q\theta)$, and $\cos^m(k^p\theta)\cos^{n-m}(k^q\theta)$ for $p = 0$, $q = 1$, $n = 10$, and $m = 4$.

Therefore, it is suggested that $\Psi_{XX}(u,v) \neq \Psi_{YY}(u,v) \neq \Psi_{XY}(u,v)$ for $q \neq p$ from numerical simulation. This also implies that three types of two-dimensional Bessel functions are not equivalent;

$$J^{p,q}_{cc}(u,v) = \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{-1}(u\cos(p\theta)+v\cos(q\theta))}\mathrm{d}\theta, \tag{72}$$

$$J^{p,q}_{sc}(u,v) = \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{-1}(u\sin(p\theta)+v\cos(q\theta))}\mathrm{d}\theta, \tag{73}$$

$$J^{p,q}_{ss}(u,v) = \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{-1}(u\sin(p\theta)+v\sin(q\theta))}\mathrm{d}\theta. \tag{74}$$

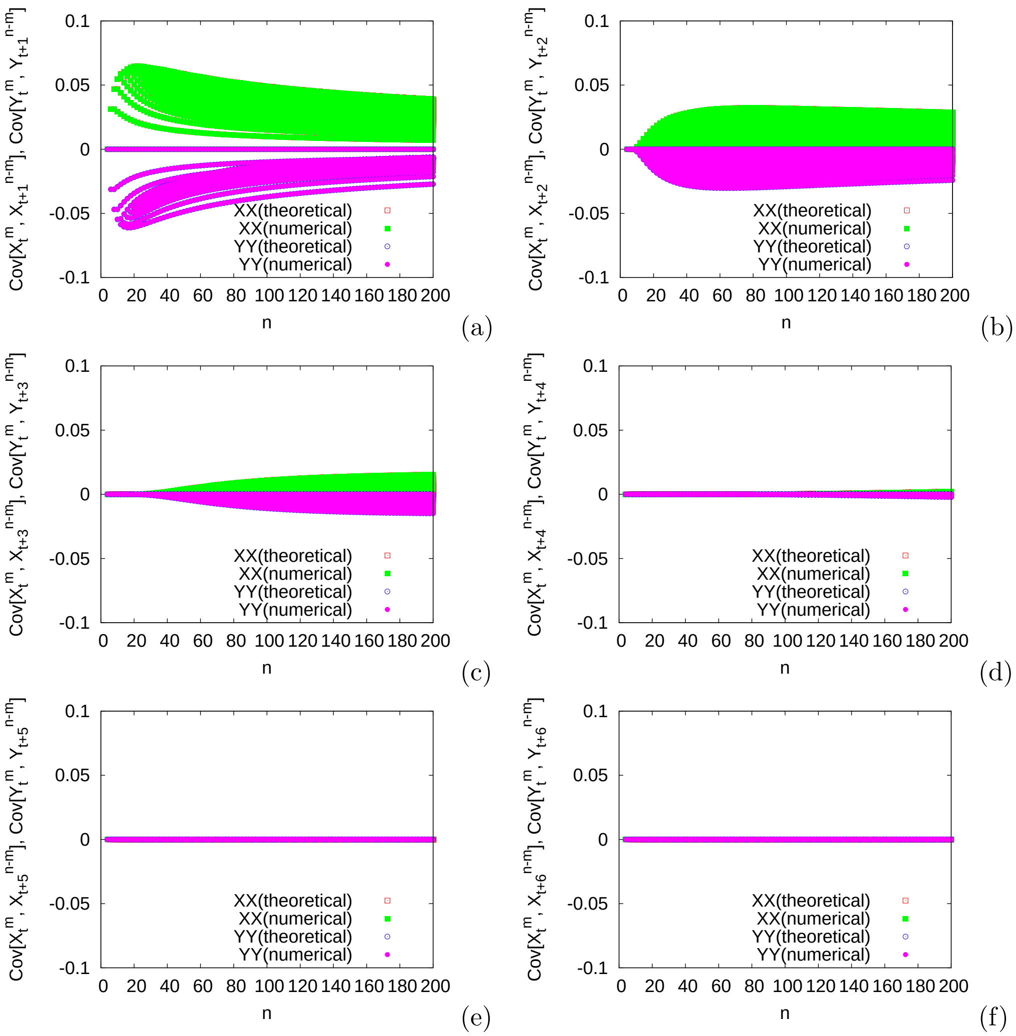

Figure 5: Scatter plots of $\mathrm{Cov}[X^m_{t+p}, X^{n-m}_{t+q}]$ and $\mathrm{Cov}[Y^m_{t+p}, Y^{n-m}_{t+q}]$ in terms of n at $k = 2$ and $p = 0$, (a) $q = 1$, (b) $q = 2$, (c) $q = 3$, (d) $q = 4$, (e) $q = 5$, and (f) $q = 6$. Unfilled squares represent theoretical values of $\mathrm{Cov}[X^m_{t+p}, X^{n-m}_{t+p}]$, filled squares numerical values of $\mathrm{Cov}[X^m_{t+p}, X^{n-m}_{t+p}]$, unfilled circles theoretical values of $\mathrm{Cov}[Y^m_{t+p}, Y^{n-m}_{t+p}]$, and filled circles numerical values of $\mathrm{Cov}[Y^m_{t+p}, Y^{n-m}_{t+p}]$.

6 Conclusion

We studied two-dimensional chaotic maps on the unit circle, which is an extension of the Chebyshev maps to two-dimensional map on the unit circle. We examined correlational properties of this two-dimensional chaotic map. We gave analytical forms of higher-order moments. Furthermore, we derived the characteristic function of both simultaneous and lagged ergodic densities. We found that these characteristic functions are given by three types of two-dimensional Bessel functions. We proved four theorems and proposed two conjectures as follows:

Theorems:

1. The higher-order covariances between x_t and y_t shows non-positive values for integers n and $m \quad (0 \leq m \leq n)$:

$$\mathrm{Cov}[X^m, Y^{n-m}] \leq 0. \tag{75}$$

2. The higher-order covariance between x_t and x_t shows non-negative values for integer n and $m \quad (0 \leq m \leq n)$:

$$\mathrm{Cov}[X^m, X^{n-m}] \geq 0. \tag{76}$$

3. The higher-order covariance between y_t and y_t shows non-negative values for n and $m \quad (0 \leq m \leq n)$:

$$\mathrm{Cov}[Y^m, Y^{n-m}] \geq 0. \tag{77}$$

4. The higher-order covariance between x_{t+p} and x_{t+q} $(p \neq q)$ shows non-negative values for integer n and $m \quad (0 \leq m \leq n)$:

$$\mathrm{Cov}[X^m_{t+p}, X^{n-m}_{t+q}] \geq 0. \tag{78}$$

Conjectures:

1. The higher-order covariances between x_{t+p} and y_{t+q} $(p \neq q)$ shows non-positive values for integers n and $m \quad (0 \leq m \leq n)$:

$$\mathrm{Cov}[X^m_{t+p}, Y^{n-m}_{t+q}] \leq 0. \tag{79}$$

2. The higher-order covariance between y_{t+p} and y_{t+q} $(p \neq q)$ shows non-positive values for n and $m \quad (0 \leq m \leq n)$:

$$\mathrm{Cov}[Y^m_{t+p}, Y^{n-m}_{t+q}] \leq 0. \tag{80}$$

Therefore, we can generate antithetic sequences as $x_0, y_0, x_1, y_1, \ldots, x_t, y_t, \ldots$ or $y_0, y_1, y_2, \ldots, y_t, \ldots$ obtained from Eq. (4). Asymmetric features between cosine and sine functions were elucidated. Using the proposed two-dimensional chaotic map, we can generate antithetic pseudo random sequences for Monte Carlo integration.

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